A QUADRATIC CREMONA TRANSFORMATION DEFINED BY A CONIC.

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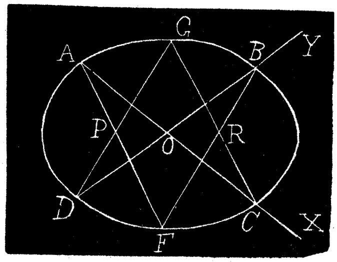
On an arbitrary conic choose four fixed points A, B, C, D. To each point P of the plane there corresponds a definite point R defined by the following construction: Join PA and PD cutting the conic again in F and G respectively. Then R is the intersection of BF with CG.

Let the equation of the base conic referred to the axes AC and BD have the general form $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0\ldots.(1)$.

We may express the equation of the conic in terms of its four intercepts a, b, c, d on the axes and an unknown parameter. Since the points $A(-a, 0)$, $B(0, b)$, $C(c, 0)$, $D(0, -d)$ lie on the conic, we obtain the relations:

$$Aa^2-2Ga+C=0$$
$$Bb^2+2Fb+C=0$$
$$Ac^2+2Gc+C=0$$
$$Bd^2-2Fd+C=0.$$



From the first and third, $2G=A(a-c)$; from the second and fourth, $2F=-B(b-d)$. Hence $C=-Aac=-Bbd$. Substituting in (1); dividing by A, and writing $h=\frac{2H}{A}$,

$bdx^2+hbdxy+acy^2+bd(a-c)x+ac(d-b)y-abcd=0\ldots.(2)$. The discriminant of (2), $\frac{-ac}{4b^2d^2}\{ab^2c+bc^2d+cd^2a+da^2b+bdh(a-c)(b-d)-h^2b^2d^2\}$, will be 0 only when $h=\frac{ab+cd}{bd}$ or $h=\frac{-(ad+bc)}{bd}$. In the former case, the conic (2) becomes $(bx+cy-bc)(dx+ay+ad)=0$, representing the straight lines AD and BC. We reject this trivial case, since to every point in the plane there corresponds the line BC. The quantity $ab+cd-hbd$ occurs below repeatedly.

For the case $h=\frac{-(ad+bc)}{bd}$, (2) becomes $(bx-ay+ab)(dx-cy-cd)=0$, representing the straight lines AB and DC. This case is not in the least trivial.

To find the co-ordinates x_1, y_1 *of the point* R, *corresponding to a point* $P(x', y')$.

The equation to AP is $y'(x+a)=y(x'+a)$; that to DP is $x(y'+d)=x'(y+d)$. The co-ordinates of the second point of intersection F of AP with the conic (2) are:

$$\frac{c\{bd(x'+a)^2+ay'(b-d)(x'+a)-a^2y'^2\}}{bd(x'+a)^2+hbdy'(x'+a)+acy'^2},$$

$$\frac{y'\{ay'(bc-cd+hbd)+bd(a+c)(x'+a)\}}{bd(x'+a)^2+hbdy'(x'+a)+acy'^2}.$$

The co-ordinates of G, the second intersection of DP with the conic (2) are:

$$\frac{x'\{x'bd(c-a+dh)+ac(b+d)(y'+d)\}}{bdx'^2+bdhx'(y'+d)+ac(y'+d)^2},$$

$$\frac{-b\{d^2x'^2+dx'(a-c)(y'+d)+ac(y'+d)^2\}}{bdx'^2+bdhx'(y'+d)+ac(y'+d)^2}.$$

The equation to BF is

$$\frac{x}{y-b}=\frac{-c(ay'+dx'+ad)(ay'-bx'-ab)}{d\{bx'+(hb-c)y'+ab\}\{ay'-bx'-ab\}}=\frac{-c(ay'+dx'+ad)}{d\{bx'+(hb-c)y'+ab\}},$$

$$\text{say}=\frac{A}{B}.$$

The equation to CG is

$$\frac{x-c}{y}=\frac{\{dx'-cy'-cd\}\{b(a-dh)x'-acy'-acd\}}{b(dx'-cy'-cd)(dx'+ay'+ad)}$$

$$=\frac{b(a-dh)x'-acy'-acd}{b(dx'+ay'+ad)},\quad \text{say}=\frac{D}{E}.$$

By elimination, the co-ordinates of R are:

$$x_1=\frac{A(bD+cE)}{AE-BD},\quad y_1=\frac{E(bA+cB)}{AE-BD}.$$

Now $AE-BD$ expanded gives $-(ab+cd-bdh)(bdx'^2+acy'^2+bdhx'y'+abdx'+acdy')$; $bD+cE=bx'(ab+cd-bdh)$; $bA+cB=-cy'(ab+cd-bdh)$.

$$\therefore x_1=\frac{bcx'(dx'+ay'+ad)}{bdx'^2+acy'^2+bdhx'y'+abdx'+acdy'},$$

$$y_1=\frac{bcy'(dx'+ay'+ad)}{bdx'^2+acy'^2+\ldots\ldots}\ldots.(3).$$

Henec $\frac{x_1}{y_1}=\frac{x'}{y'}$.....(4), or PR always passes through O, Pascal's Theorem for a hexagon inscribed in a conic.

Solving (3) for x' and y',

$$x'=\frac{adx_1(-bx_1-cy_1+bc}{bdx_1^2+acy_1^2+bdhx_1y_1-abcy_1-bcdx_1},\quad y'=\frac{ady_1(-bx_1-cy_1+bc)}{bdx_1^2+\ldots.}\ldots.(5).$$

The reciprocity between P and R shown by (3) and (5) is evident geometrically.

If P describes a straight line $y=mx+l$, the locus of R is a conic.

Substituting the value (5) in $y'=mx'+l$, and dropping the subscripts to the co-ordinates of R, we find its locus: $bdx^2(l-ma)+dxy(ab-mac+blh)+acy^2(l+d)-abcy(l+d)-bcdx(l-ma)=0$....(6).

Its discriminant is $-\frac{1}{4}ab^2c^{\prime\prime}dl(l+d)(l-ma)(ab+cd-bdh)$. Rejecting as before the trivial case $h=\frac{ab+cd}{bd}$, this can be zero only if $l=0,-d$, or ma.

Hence (6) will degenerate to a pair of straight lines in just three cases:

If $l=0$, (6) becomes $ad(y-mx)(bx+cy-bc)=0$. Hence if P describes a straight line through O, the locus of R is this same line through O and the line BC, given when P is at O as the indeterminate intersection of BC with itself.

If $l=-d$, (6) becomes $-x\{bdx(ma+d)+dy(mac-ab+bdh)$ $-bcd(ma+d)\}=0$. The second factor gives the equation to the line through C and the second intersection of $y=mx-d$ with the base conic (2). Hence the transform of any line DG through D is CG and the y-axis BD, the latter being given when P is at D as the intersection of BD with an indeterminate line through C.

If $l=ma$, (6) becomes $y\{adx(b-mc+mbh)+acy(d+ma)$ $-abc(d+ma)\}=0$. The second factor gives the equation to the line through B and the second intersection of $y=m(x+a)$ with the base conic (2). Hence the transform of any line AF through A is BF and the x-axis AC, the latter being given when P is at A as the intersection of AC with an indeterminate line through B.

The conic (6) passes through the points O, B, C, and, since every point on the base conic is self corresponding, the points in which $y=mx+l$ intersects the base conic.

The line BC whose equation is $bx+cy-bc=0$ transforms into $b^2dx^2(a+c)+ac^2y^2(b+d)+bcdxy(2a+bh)-b^2cdx(a+c)-abc^2y(b+d)=0$ .....(7). The tangent to it at the origin, $bdx(a+c)+acy(b+d)=0$, passes through the intersection $\left(\frac{ac(b+d)}{ab-dc}, \frac{-bd(a+c)}{ab-dc}\right)$ of AD and BC. Further, (7) is tangent to the base conic (2) at the points B and C. Thus, the tangent at B to either (2) or (7) is $bdx(a-c+bh)+acy(b+d)-abc(b+d)=0$.

Applying (3), the equation to the curve which transforms *into* the line AD, or $dx+ay+ad=0$, is $bd^2x^2(a+c)+a^2cy^2(b+d)+abdxy(2c+dh)$ $+abd^2x(a+c)+a^2cdy(b+d)=0$, which has the same tangent at the origin as (7).

The line at infinity transforms by (5) into the conic bdx^2+acy^2 $+bdhxy-bcdx-abcy=0$....(8). Since (2) and (8) differ only by the linear expresssion $ad(bx+cy-bc)$, the points of intersection of BC with (2) lie on (8); also (2) and (8) are simultaneously ellipses, parabolas, or hyperbolas. The discriminant of (8), $-\frac{1}{4}ab^2c^2d(ab+cd-bdh)$, shows that it breaks up into two right lines only in trivial case above excluded.

The conic which transforms into the line at infinity, given by the vanishing of the denominator of (3), is $bdx^2+acy^2+bdhxy+abdx+acdy=0$....(9) passing through O, A, D. Subtracting (8) from (9), we find their intersections lie on the line $bdx(a+c)+acy(b+d)=0$, which passes through K, the intersection of AD and BC.

Note that the conics (2), (8) and (9) are *similar*

Generally, *the intersections of the conic which transforms into any straight line with the conic into which that straight line transforms lie two on the straight line OK and two on the line itself, the latter two being real and dis-*

tinct, coincident, or imaginary, according as the line intersects the base conic in two real, coincident, or imaginary points.

The conic which transforms into $y=mx+l$, given by substituting from (3) into $y_1=mx_1+l$, is $bdx^2(l+mc)+acy^2(l-b)+bxy(mac-cd+ldh)$
$+abdx(l+mc)+acdy(l-b)=0\ldots.(10)$.

This intersects the conic (6) into which $y=mx+l$ transforms in four points, which, if we subtract (6) from (10) and factor, are seen to lie on

$$\{acy(b+d)+bdx(a+c)\}\{mx-y+l\}=0.$$

O is one of the two intersections lying on OK. Call the other H. Then the point in which $y=mx+l$ meets OK and the point H mutually correspond. *We thus have an involution marked out on OK.*

We saw above that the points A, D, O transform into the lines AC, DB, BC respectively. Now we can prove either geometrically or analytically that the lines AD, AO, DO transform into the points O, C, B respectively. Thus the sides and vertices of $\triangle ADO$ transform into the vertices and sides of $\triangle OBC$. With this exception the correspondence between the points in the two systems is one to one. The *projective* treatment of this transformation and its dual will be given elsewhere.

[For Projective Treatment, see my paper in the *Rendiconti del Circolo di Palermo.*]

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

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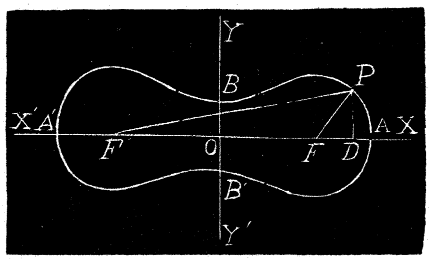
The Cassinian Oval is the locus of a point the *product* of whose distances from two fixed points is constant.



Let P be any point on the curve, F and F' the foci, O the middle point of FF', $OD=x$, $DP=y$, $OF=c$, $FP=\rho$, and $F'P=\rho'$; then, according to the definition of the curve, $\rho\rho'=m^2\ldots.(1)$. From the diagram, $\rho=\pm\sqrt{[(x-c)^2+y^2]}$ and $\rho'=\pm\sqrt{[(x+c)^2+y^2]}$; that is, from (1) we obtain the equation.

$$\sqrt{\{[(x-c)^2+y^2]\times[(x+c)^2+y^2]\}}=m^2\ldots.(a).$$

$\therefore (x^2+y^2+c^2)^2-4c^2x^2=m^4\ldots.(2)$; and this is the Cartesian equation of the Cassinian Oval, the co-ordinate axes being rectangular.

Put $OP=r$, and the $\angle POD=(\frac{1}{2}\pi-\theta)$; then $OD=x=r\sin\theta$, and $PD=y=r\cos\theta$. From (2), therefore, we have $r^4+2c^2(1-2\sin^2\theta)r^2=m^4-c^4\ldots.(b)$, or $r^4+(2c^2\cos 2\theta)r^2=m^4-c^4\ldots.(3)$, which is a convenient